A VIEW OF MATHEMATICS

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Mathematics is the backbone of modern science and a remarkably efficient source of new concepts and tools to understand the "reality" in which we participate.
It plays a basic role in the great new theories of physics of the XXth century such as general relativity, and quantum mechanics.

The nature and inner workings of this mental activity are often misunderstood or simply ignored even among scientists of other disciplines. They usually only make use of rudimentary mathematical tools that were already known in the XIXth century and miss completely the strength and depth of the constant evolution of our mathematical concepts and tools.

I was asked to write a general introduction on Mathematics which I ended up doing from a rather personal point of view rather than producing the usual endless litany "X did this and Y did that". The evolution of the concept of "space" in mathematics serves as a unifying theme starting from some of its historical roots and going towards more recent developments in which I have been more or less directly involved.

Contents

1. The Unity of Mathematics

It might be tempting at first to view mathematics as the union of separate parts such as Geometry, Algebra, Analysis, Number theory etc... where the first is dominated by the understanding of the concept of “space”, the second by the art of manipulating “symbols”, the next by the access to “infinity” and the “continuum” etc...

This however does not do justice to one of the most essential features of the mathematical world, namely that it is virtually impossible to isolate any of the above parts from the others without depriving them from their essence. In that way the corpus of mathematics does resemble a biological entity which can only survive as a whole and would perish if separated into disjoint pieces.

The first embryo of mental picture of the mathematical world one can start from is that of a network of bewildering complexity between basic concepts. These basic concepts themselves are quite simple and are the result of a long process of “distillation” in the alembic of the human thought.

Where a dictionary proceeds in a circular manner, defining a word by reference to another, the basic concepts of mathematics are infinitely closer to an “indecomposable element”, a kind of “elementary particle” of thought with a minimal amount of ambiguity in their definition.

This is so for instance for the natural numbers where the number 3 stands for that quality which is common to all sets with three elements. That means sets which become empty exactly after we remove one of its elements, then remove another and then remove another. In that way it becomes independent of the symbol 3 which is just a useful device to encode the number.

Whereas the letters we use to encode numbers are dependent of the sociological and historical accidents that are behind the evolution of any language, the mathematical concept of number and even the specificity of a particular number such as 17 are totally independent of these accidents.

The “purity” of this simplest mathematical concept has been used by Hans Freudenthal to design a language for cosmic communication which he called “Lincos” [39].

The scientific life of mathematicians can be pictured as a trip inside the geography of the “mathematical reality” which they unveil gradually in their own private mental frame.

It often begins by an act of rebellion with respect to the existing dogmatic description of that reality that one will find in existing books. The young “to be mathematician” realize in their own mind that their perception of the mathematical world captures some features which do not quite fit with the existing dogma. This first act is often due in most cases to ignorance but it allows one to free oneself from the reverence to authority by relying on one’s intuition provided it is backed up by actual proofs. Once mathematicians get to really know, in an original and “personal” manner, a small part of the mathematical world, as esoteric as it can look at first[1], their trip can really start. It is of course vital all along not to break the “fil d’arianne” which allows to constantly keep a fresh eye on whatever one

[1] my starting point was localization of roots of polynomials.

will encounter along the way, and also to go back to the source if one feels lost at times...
It is also vital to always keep moving. The risk otherwise is to confine oneself in a relatively small area of extreme technical specialization, thus shrinking one's perception of the mathematical world and of its bewildering diversity.

The really *fundamental point* in that respect is that while so many mathematicians have been spending their entire scientific life exploring that world they all agree on its contours and on its connexity: whatever the origin of one's itinerary, one day or another if one walks long enough, one is bound to reach a well known town *i.e.* for instance to meet elliptic functions, modular forms, zeta functions. "All roads lead to Rome" and the mathematical world is "connected".

In other words there is just "one" mathematical world, whose exploration is the task of all mathematicians and they are all in the same boat somehow.

Moreover exactly as the existence of the external material reality seems undeniable but is in fact only justified by the coherence and consensus of our perceptions, the existence of the mathematical reality stems from its coherence and from the consensus of the findings of mathematicians. The fact that proofs are a necessary ingredient of a mathematical theory implies a much more reliable form of "consensus" than in many other intellectual or scientific disciplines. It has so far been strong enough to avoid the formation of large gatherings of researchers around some "religious like" scientific dogma imposed with sociological imperialism.

Most mathematicians adopt a pragmatic attitude and see themselves as the explorers of this "mathematical world" whose existence they don't have any wish to question, and whose structure they uncover by a mixture of intuition, not so foreign from "poetical desire"[2], and of a great deal of rationality requiring intense periods of concentration.

Each generation builds a "mental picture" of their own understanding of this world and constructs more and more penetrating mental tools to explore previously hidden aspects of that reality.

Where things get really interesting is when unexpected bridges emerge between parts of the mathematical world that were previously believed to be very far remote from each other in the natural mental picture that a generation had elaborated. At that point one gets the feeling that a sudden wind has blown out the fog that was hiding parts of a beautiful landscape.

I shall describe at the end of this paper one recent instance of such a bridge. Before doing that I'll take the concept of "space" as a guide line to take the reader through a guided tour leading to the edge of the actual evolution of this concept both in algebraic geometry and in noncommutative geometry. I shall also review some of the "fundamental" tools that are at our disposal nowadays such as "positivity", "cohomology", "calculus", "abelian categories" and most of all "symmetries" which will be a recurrent theme in the three different parts of this text.

It is clearly impossible to give a "panorama" of the whole of mathematics in a reasonable amount of write up. But it is perfectly possible, by choosing a precise

[2]as emphasised by the French poet Paul Valery.

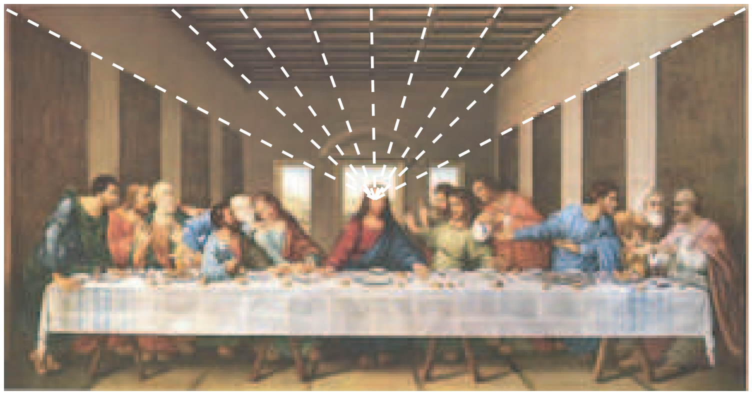

Figure 1. Perspective

theme, to show the frontier of certain fundamental concepts which play a central role in mathematics and are still actively evolving.

The concept of "space" is sufficiently versatile to be an ideal theme to display this active evolution and we shall confront the mathematical concept of space with physics and more precisely with what Quantum Field Theory teaches us and try to explain several of the open questions and recent findings in this area.

2. The concept of Space

The mental pictures of geometry are easy to create by exploiting the visual areas of the brain. It would be naive however to believe that the concept of "space" *i.e.* the stage where the geometrical shapes develop, is a straightforward one. In fact as we shall see below this concept of "space" is still undergoing a drastic evolution.

The Cartesian frame allows one to encode a point of the euclidean plane or space by two (or three) real numbers $x^\mu \in \mathbb{R}$. This irruption of "numbers" in geometry appears at first as an act of violence undergone by geometry thought of as a synthetic mental construct.

This "act of violence" inaugurates the duality between geometry and algebra, between the eye of the geometor and the computations of the algebraists, which run in time contrasting with the immediate perception of the visual intuition.

Far from being a sterile opposition this duality becomes extremely fecund when geometry and algebra become allies to explore unknown lands as in the new algebraic geometry of the second half of the twentieth century or as in noncommutative geometry, two existing frontiers for the notion of space.

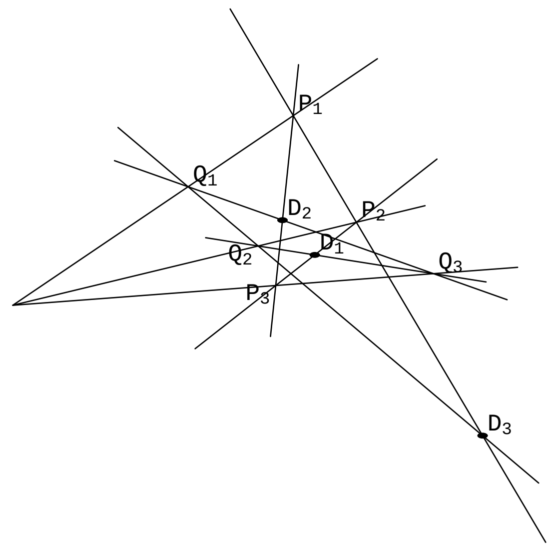

FIGURE 2. Desargues's Theorem : *Let* P_j *and* Q_j, $j \in \{1,2,3\}$ *be points such that the three lines* (P_j, Q_j) *meet. Then the three points* $D_j := (P_k, P_l) \cap (Q_k, Q_l)$ *are on the same line.*

2.1. **Projective geometry.**

Let us first briefly describe projective geometry a telling example of the above duality between geometry and algebra.

In the middle of the XVIIth century, G. Desargues, trying to give a mathematical foundation to the methods of perspective used by painters and architects founded real projective geometry. The real projective plane of Desargues is the set $P_2(\mathbb{R})$ of lines through the origin in three space $\mathbb{R}^3$. This adds to the usual points of the plane a "line at infinity" which gives a perfect formulation and support for the empirical techniques of perspective.

In fact Desargues's theorem (figure 2) can be viewed as the base for the axiomatization of projective geometry.

This theorem is a consequence of the extremely simple four axioms which define projective geometry, but it requires for its proof that the dimension of the geometry be strictly larger than two.

These axioms express the properties of the relation "$P \in L$" *i.e.* the point P belongs to the line L, they are:

- Existence and uniqueness of the straight line containing two distinct points.
- Two lines defined by four points located on two meeting lines actually meet in one point.
- Every line contains at least three points.
- There exists a finite set of points that generate the whole geometry by iterating the operation passing from two points to all points of the line they span.

In dimension $n = 2$, Desargues's theorem is no longer a consequence of the above axioms and one has to add it to the above four axioms. The Desarguian geometries of dimension n are exactly the projective spaces $P_n(K)$ of a (not necessarily commutative) field K.

They are in this way in perfect duality with the key concept of algebra: that of field.

What is a field ? It is a set of "numbers" that one can add, multiply and in which any non-zero element has an inverse so that all familiar rules[3] are valid. One basic example is given by the field $\mathbb{Q}$ of rational numbers but there are many others such as the field $\mathbb{F}_2$ with two elements or the field $\mathbb{C}$ of complex numbers. The field $\mathbb{H}$ of quaternions of Hamilton is a beautiful example of non-commutative field.

Complex projective geometry *i.e.* that of $P_n(\mathbb{C})$ took its definitive form in "La Géométrie" of Monge in 1795. The presence of complex points on the side of the real ones simplifies considerably the overall picture and gives a rare harmony to the general theory by the simplicity and generality of the results. For instance all circles of the plane pass through the "cyclic points" a pair of points (introduced by Poncelet) located on the line at infinity and having complex coordinates. Thus as two arbitrary conics any pair of circles actually meet in four points, a statement clearly false in the real plane.

The need for introducing and using complex numbers even to settle problems whose formulation is purely "real" had already appeared in the XVIth century for the resolution of the third degree equation. Indeed even when the three roots of such an equation are real the conceptual form of these roots in terms of radicals necessarily passes through complex numbers. (*cf.* Chapters 11 to 23 in Cardano's book of 1545 *Ars magna sive de regulis algebraicis*).

2.2. **The Angel of Geometry and the Devil of Algebra.**

The duality

$$\underline{\text{Geometry}} \quad | \quad \underline{\text{Algebra}} \tag{1}$$

already present in the above discussion of projective geometry allows, when it is viewed as a mutual enhancement, to translate back and forth from geometry to algebra and obtain statements that would be hard to guess if one would stay confined in one of the two domains. This is best illustrated by a very simple example.

The geometric result, due to Frank Morley, deals with planar geometry and is one of the few results about the geometry of triangles that was apparently unknown to Greek mathematicians. You start with an arbitrary triangle ABC and trisect each angle, then you consider the intersection of consecutive trisectors, and obtain another triangle $\alpha\beta\gamma$ (fig.3). Now Morley's theorem, which he found around 1899, says that *whichever triangle ABC you start from, the triangle $\alpha\beta\gamma$ is always equilateral.*

Here now is an algebraic "transcription" of this result. We start with an arbitrary commutative field K and take three "affine" transformations of K. These are maps g from K to K of the form $g(x) = \lambda\, x + \mu$, where $\lambda \neq 0$. Given such a transformation

[3] except possibly the commutativity $xy = y\,x$ of the product.

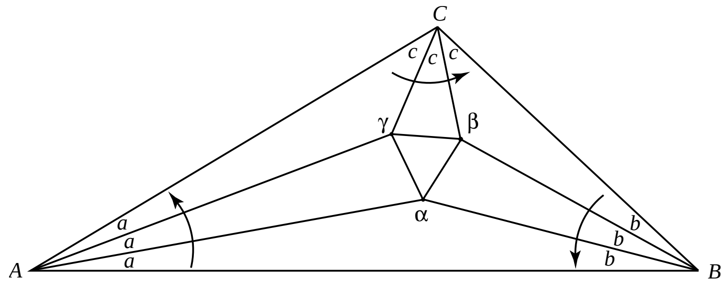

FIGURE 3. Morley's Theorem : *The triangle $\alpha\beta\gamma$ obtained from the intersection of consecutive trisectors of an arbitrary triangle ABC is always equilateral.*

the value of $\lambda \in K$ is unique and noted $\delta(g)$. For $g \in G$, $g(x) = \lambda x + \mu$ not a translation, *i.e.* $\lambda \neq 1$ one lets $\mathrm{fix}(g) = \alpha$ be the unique fixed point $g(\alpha) = \alpha$ of g. These maps form a group $G(K)$ (*cf.* subsection 2.4) called the "affine group" and the algebraic counterpart of Morley's theorem reads as follows

Let $f, g, h \in G$ be such that fg, gh, hf and fgh are not *translations and let $j = \delta(fgh)$. The following two conditions are equivalent,*

a) $f^3 g^3 h^3 = 1$.
b) $j^3 = 1$ *and* $\alpha + j\beta + j^2\gamma = 0$ *where* $\alpha = \mathrm{fix}\,(fg)$, $\beta = \mathrm{fix}\,(gh)$, $\gamma = \mathrm{fix}\,(hf)$.

This is a sufficiently general statement now, involving an arbitrary field K and its proof is a simple "verification", which is a good test of the elementary skills in "algebra".

It remains to show how it implies Morley's result. But the fundamental property of "flatness" of Euclidean geometry, namely

$$a + b + c = \pi \tag{2}$$

where a, b, c are the angles of a triangle (A, B, C) is best captured algebraically by the equality

$$F\,G\,H = 1$$

in the affine group $G(\mathbb{C})$ of the field $K = \mathbb{C}$ of complex numbers, where F is the rotation of center A and angle $2a$ and similarly for G and H. Thus if we let f be the rotation of center A and angle $2a/3$ and similarly for g and h we get the condition $f^3\,g^3\,h^3 = 1$.

The above equivalence thus shows that $\alpha + j\beta + j^2\gamma = 0$, where α, β, γ, are the fixed points of fg, gh et hf and where $j = \delta(fgh)$ is a non-trivial cubic root of unity. The relation $\alpha + j\beta + j^2\gamma = 0$ is a well-known characterization of equilateral triangles (it means $\frac{\alpha-\beta}{\gamma-\beta} = -j^2$, so that one passes from the vector $\overrightarrow{\beta\gamma}$ to $\overrightarrow{\beta\alpha}$ by a rotation of angle $\pi/3$).

Finally it is easy to check that the fixed point α, $f\big(g(\alpha)\big) = \alpha$ is the intersection of the trisectors from A and B closest to the side AB. Indeed the rotation g moves it to its symmetric relative to AB, and f puts it back in place. Thus we proved that the triangle (α, β, γ) is equilateral. In fact we also get for free 18 equilateral triangles obtained by picking other solutions of $f^3 = F$ etc...

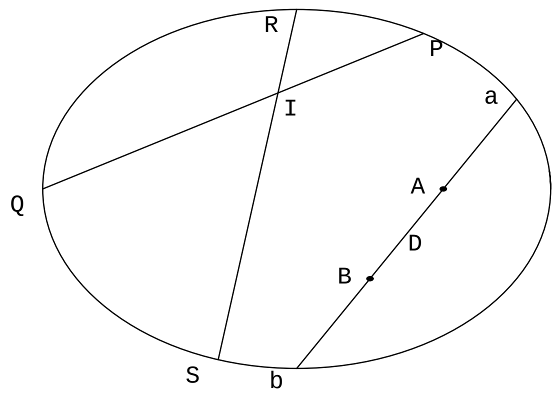

FIGURE 4. Klein model

This is typical of the power of the duality between on the one hand the visual perception (where the geometrical facts can be sort of obvious) and on the other hand the algebraic understanding. Then, provided one can write things in algebraic terms, one enhances their power and makes them applicable in totally different circumstances. For instance the above theorem holds for a finite field, it holds for instance for the field F_4 which has cubic roots of unity.... So somehow, passing from the geometrical intuition to the algebraic formulation allows one to increase the power of the original "obvious" fact, a bit like language can enhance the strength of perception, in using the "right words".

2.3. **Noneuclidean geometry.**

The discovery of Noneuclidean geometry at the beginning of the XIXth century frees the geometric concepts whose framework opens up in two different directions.

- The first opening is intimately related to the notion of symmetry and to the theory of Lie groups.
- The second is the birth of the geometry of curved spaces of Gauss and Riemann, which was to play a crucial role soon afterwards in the elaboration of general relativity by Einstein.

A particularly simple model of noneuclidean geometry is the Klein model. The points of the geometry are those points of the plane which are located inside a fixed ellipse E (*cf.* Fig. 4). The lines of the geometry are the intersections of ordinary euclidean lines with the inside of the ellipse.

The fifth postulate of Euclid on 'flatness" *i.e.* on the sum of the angles of a triangle (2) can be reformulated as the uniqueness of the line parallel[4] to a given line D passing through a point $I \notin D$. In this form this postulate is thus obviously violated in the Klein model since through a point such as I pass several lines such as $L = PQ$ and $L' = RS$ which do not intersect D.

It is however not enough to give the points and the lines of the geometry to determine it in full. One needs in fact also to specify the relations of "congruence"

[4] *i.e.* not intersecting.

between two segments[5] AB and CD. The congruence of segments means that they have the same "length" and the latter is specified in the Klein model by

$$\text{length}(AB) = \log\left(\text{cross ratio}(A, B; b, a)\right) \tag{3}$$

where the cross-ratio of four points P_j on the same line with coordinates s_j is by definition

$$\text{cross ratio}(P_1, P_2; P_3, P_4) = \frac{(s_1 - s_3)(s_2 - s_4)}{(s_2 - s_3)(s_1 - s_4)} \tag{4}$$

Noneuclidean geometry was discovered at the beginning of the XIXth century by Lobachevski and Bolyai, after many efforts by great mathematicians such as Legendre to show that the fifth Euclid's axiom was unnecessary. Gauss discovered it independently and did not make his discovery public, but by developing the idea of "intrinsic curvature" he was already ways ahead anyway.

All of Euclid's axioms are fulfilled by this geometry[6] except for the fifth one. It is striking to see, looking back, the fecundity of the question of the independence of the fifth axiom, a question which at first could have been hastily discarded as a kind of mental perversion in trying to eliminate one of the axioms in a long list that would not even look any shorter once done.

What time has shown is that far from just being an esoteric counterexample Noneuclidean geometry is of a rare richness and fecundity. By breaking the traditional framework it generated two conceptual openings which we alluded to above and that will be discussed below, starting from the S. Lie approach.

2.4. **Symmetries.**

One way to define the congruence of segments in the above Klein model, without referring to "length", *i.e.* to formula (3), is to use the natural symmetry group G of the geometry given by the projective transformations T of the plane that preserve the ellipse E. Then by definition, two segments AB and CD are congruent if and only if there exists such a transformation $T \in G$ with $T(A) = C$, $T(B) = D$.

The set of these transformations forms a group *i.e.* one can compose such transformations and obtain another one, *i.e.* one has a "law of composition"

$$(S, T) \to S \circ T \in G\,, \quad \forall S, T \in G\,, \tag{5}$$

of elements of G in which multiple products are defined independently of the parenthesis, *i.e.*

$$(S \circ T) \circ U = S \circ (T \circ U)) \tag{6}$$

a condition known as "associativity", while the identity transformation id fulfills

$$S \circ \text{id} = \text{id} \circ S = S \tag{7}$$

and every element S of the group admits an inverse, uniquely determined by

$$S \circ S^{-1} = S^{-1} \circ S = \text{id} \tag{8}$$

Group theory really took off with the work of Abel and Galois on the resolution of polynomial equations (*cf.* section 3.6). In that case the groups involved are finite

[5] as well as between angles.

[6] These Euclid's axioms are notably more complicated than those of projective geometry mentioned above.

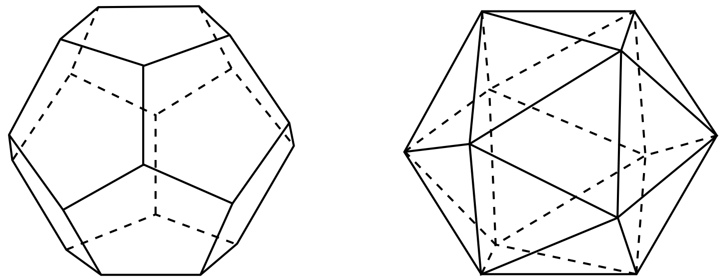

Figure 5. Dodecahedron and Icosahedron

groups *i.e.* finite sets G endowed with a law of composition fulfilling the above axioms. Exactly as an integer can be prime *i.e.* fail to have non-trivial divisors a finite group can be "simple" *i.e.* fail to map surjectively to a smaller non-trivial group while respecting the composition rule.

The classification of all finite simple groups is one of the great achievements of XXth century mathematics.

The group of symmetries of the above Klein geometry is not finite since specifying one of these geometric transformations involves in fact choosing three continuous parameters. It falls under the theory of S. Lie which was in fact a direct continuation of the ideas formulated by Galois.

These ideas of Sophus Lie were reformulated in the "Erlangen program" of Félix Klein and successfully developed by Elie Cartan whose classification of Lie groups is another great success of XXth century mathematics. Through the work of Chevalley on algebraic groups the theory of Lie groups played a key role in the classification of finite simple groups.

2.5. **Line element and Riemannian geometry.**

The congruence of segments in noneuclidean geometry can also be defined in terms of the equality ot their "length" according to (3). In fact building on Gauss's discovery of the intrinsic geometry of surfaces, Riemann was able to extend geometry far beyond the spaces which admit enough symmetries to move around rigid bodies and allow one to define the congruence of segments in terms of symmetries.

He considered far more general spaces in which one cannot (in general) move a geometric shape such as a triangle for instance without deforming it *i.e.* altering the length of some of its sides or some of its angles.

The first new geometric input is the idea of a *space* as a manifold of points of arbitrary dimension, defined in an intrinsic manner independently of any embedding in Euclidean space. In a way it is a continuation of Descartes's use of real numbers as coordinates.

This is now understood in modern language as differentiable manifolds, a notion which models the range of continuous variables of many dimensions. The simplest examples include the parameter spaces of mechanical systems, the positions of a

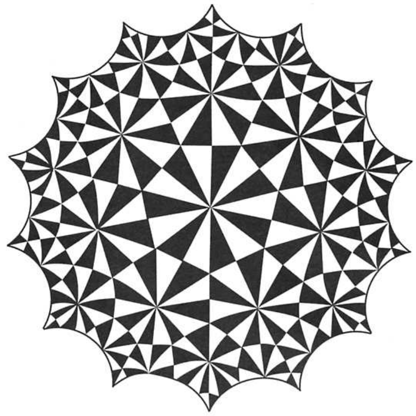

FIGURE 6. Triangles in the Poincaré model

solid body[7], etc...

The second key idea in Riemann's point of view is that whereas one cannot carry around rigid bodies one does dispose of a unit of length which can be carried around and allows one to measure length of small intervals ds. The distance $d(x,y)$ between two points x and y is then given by adding up the length of the small intervals along a path γ between x and y and then looking for the minimal such length,

$$d(x,y) = \mathrm{Inf}\{\int_\gamma ds\,|\gamma \text{ is a path between } x \text{ and } y\} \tag{9}$$

Thus the geometric data is entirely encoded by the "line element" and one assumes that its square ds^2 *i.e.* the square length of infinitesimal intervals in local coordinates x^μ around any point, is given as a quadratic form,

$$ds^2 = g_{\mu\nu} dx^\mu \, dx^\nu \tag{10}$$

For instance the length of the shorter sides of the black triangles appearing in Fig. 6 are all equal for the Riemannian metric which encodes the Poincaré model of non-euclidean geometry. The corresponding line element ds is prescribed by :

$$ds^2 = \,(R^2 - \rho^2)^{-2}\, ds_E^2 \tag{11}$$

where ds_E is the Euclidean line element and ρ is the Euclidean distance to the center of the circle of radius R whose inside forms the set of points of the Poincaré model of non-euclidean geometry.

The geometry is entirely specified by the pair (M, ds) where M is the manifold of points and where the line element ds is given by (10).

This "metric" standpoint, as compared to the study of "symmetric spaces", admits a considerable additional freedom since the choice of the $g_{\mu\nu}$ is essentially arbitrary

[7]as an amusing exercice the reader will give a lower bound for the dimension of the manifold of positions of the human body.

(except for "positivity conditions" asserting that the square length is positive, which need to be relaxed to fit with space-time physics).

One can easily understand how the decisive advantage given by this flexibility allows a direct link with the laws of physics by providing a geometric model for Newton's law in an arbitrary potential. First the notion of straight line is a concept of traditional geometry that extends most directly in Riemannian geometry under the name of "geodesics". A *geodesic* is a path γ which achieves the minimal value of (9) between any two sufficiently nearby points $x, y \in \gamma$. The calculus of variations allows one to formulate geodesics as solutions of the following differential equation, which continues to make sense in arbitrary signature of the quadratic form (10),

$$\frac{d^2 x^\mu}{dt^2} = -\frac{1}{2} g^{\mu\alpha}(g_{\alpha\nu,\rho} + g_{\alpha\rho,\nu} - g_{\nu\rho,\alpha}) \frac{dx^\nu}{dt} \frac{dx^\rho}{dt} \tag{12}$$

where the $g_{\alpha\nu,\rho}$ are the partial derivatives $\partial_{x^\rho} g_{\alpha\nu}$ of $g_{\alpha\nu}$.

It is the variability or arbitrariness in the choice of the $g_{\mu\nu}$ that prevents a general Riemannian space to be homogeneous under symmetries so that rigid motion is in general impossible, but it is that same arbitrariness that allows one to encompass, by the geodesic equation, many of the laws of mechanics which in general depend upon rather arbitrary functions such as the Newtonian potential.

Indeed one of the crucial starting points of general relativity is the identity between the geodesic equation and Newton's law of gravity in a potential V. If in the space-time Minkowski metric, which serves as a model for special relativity, one modifies the coefficient of dt^2 by adding the Newtonian potential V the geodesic equation becomes Newton's law[8]. In other words by altering not the measurement of length but that of time one can model the gravitational law as the lines in a curved space-time and express geometrically the equivalence principle as the existence of a geometric background independent of the nature of the matter that is used to test it by its inertial motion.

The Poisson law expressing the Laplacian of V from the matter distribution then becomes Einstein's equations, which involve the curvature tensor and were missed at first since the only "covariant" first derivatives of the gravitational potential $g_{\mu\nu}$ all vanish identically[9]. The simplest way to remember Einstein's equations is to derive them from an action principle and in the vacuum this is provided by the Einstein-Hilbert action, which in euclidean signature is of the form,

$$S_E[g_{\mu\nu}] = -\frac{1}{G} \int_M r \sqrt{g}\, d^4x \tag{13}$$

where G is a constant and $\sqrt{g}\, d^4x$ is the Riemannian volume form and r the scalar curvature which we shall meet again later on, in section 4.1. The Einstein's equations in the presence of matter are then readily obtained by adding the matter action, minimally coupled to the $g_{\mu\nu}$, to the above one (13).

Not only did Riemannian geometry play a basic role in the development of general relativity but it became the central paradigm in geometry in the XXth century.

[8] this holds neglecting terms of higher order.

[9] Einstein wrote a paper explaining that then there could be no fully covariant set of equations, curvature fortunately saved the day.

After Poincaré's uniformization theorem and Cartan's classification of symmetric spaces, M. Gromov has revolutionized Riemannian geometry through the power of his vision. Thurston's geometrization conjecture of three manifolds has been another great driving force behind the remarkable progresses of geometry in the recent years.

It is interesting to note that Riemann was well aware of the limits of his own point of view as he clearly expressed in the last page of his inaugural lecture [55]:

"Questions about the immeasurably large are idle questions for the explanation of Nature. But the situation is quite different with questions about the immeasurably small. Upon the exactness with which we pursue phenomenon into the infinitely small, does our knowledge of their causal connections essentially depend. The progress of recent centuries in understanding the mechanisms of Nature depends almost entirely on the exactness of construction which has become possible through the invention of the analysis of the infinite and through the simple principles discovered by Archimedes, Galileo and Newton, which modern physics makes use of. By contrast, in the natural sciences where the simple principles for such constructions are still lacking, to discover causal connections one pursues phenomenon into the spatially small, just so far as the microscope permits. Questions about the metric relations of Space in the immeasurably small are thus not idle ones.
If one assumes that bodies exist independently of position, then the curvature is everywhere constant, and it then follows from astronomical measurements that it cannot be different from zero; or at any rate its reciprocal must be an area in comparison with which the range of our telescopes can be neglected. But if such an independence of bodies from position does not exist, then one cannot draw conclusions about metric relations in the infinitely small from those in the large; at every point the curvature can have arbitrary values in three directions, provided only that the total curvature of every measurable portion of Space is not perceptibly different from zero. Still more complicated relations can occur if the line element cannot be represented, as was presupposed, by the square root of a differential expression of the second degree. Now it seems that the empirical notions on which the metric determinations of Space are based, the concept of a solid body and that of a light ray, lose their validity in the infinitely small; it is therefore quite definitely conceivable that the metric relations of Space in the infinitely small do not conform to the hypotheses of geometry; and in fact one ought to assume this as soon as it permits a simpler way of explaining phenomena.
The question of the validity of the hypotheses of geometry in the infinitely small is connected with the question of the basis for the metric relations of space. In connection with this question, which may indeed still be ranked as part of the study of Space, the above remark is applicable, that in a discrete manifold the principle of metric relations is already contained in the concept of the manifold, but in a continuous one it must come from something else. Therefore, either the reality underlying Space must form a discrete manifold, or the basis for the metric relations must be sought outside it, in binding forces acting upon it.
An answer to these questions can be found only by starting from that conception of phenomena which has hitherto been approved by experience, for which Newton laid the foundation, and gradually modifying it under the compulsion of facts which cannot be explained by it. Investigations like the one just made, which begin from general concepts, can serve only to insure that this work is not hindered by too restricted concepts, and that progress in comprehending the connection of things is not obstructed by traditional prejudices.
This leads us away into the domain of another science, the realm of physics, into which the nature of the present occasion does not allow us to enter".

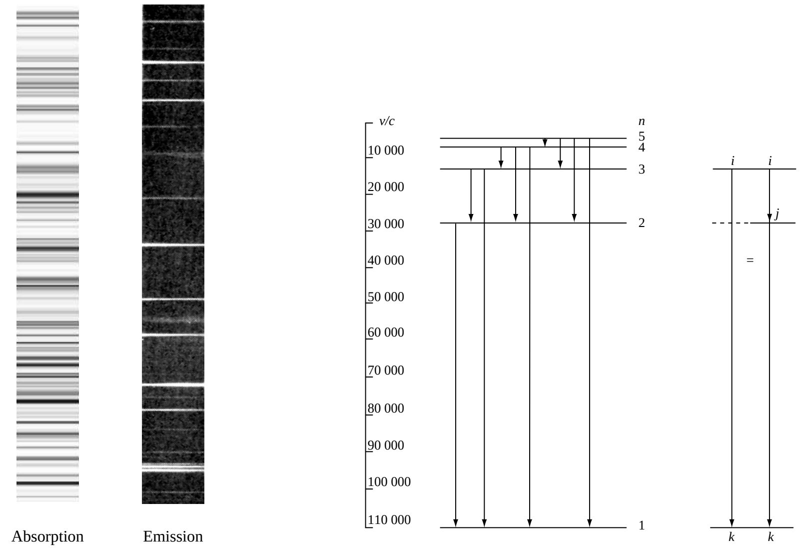

FIGURE 7. Spectra and Ritz-Rydberg law

Of course Riemann could not[10] anticipate on the other major discovery of physics in the XXth namely quantum mechanics which started in 1900 with Planck's law. This discovery as we shall explain now, called for an extension of Riemann's ideas to spaces of a wilder type than ordinary manifolds.

2.6. **Noncommutative geometry.**

The first examples of such "new" spaces came from the discovery of the quantum nature of the phase space of the microscopic mechanical system describing an atom. Such a system manifests itself through its interaction with radiation and the corresponding spectra (Fig. 7). The basic laws of spectroscopy, as found in particular by Ritz and Rydberg (Fig. 7), are in contradiction with the "classical manifold" picture of the phase space and Heisenberg was the first to understand that for a microscopic mechanical system the coordinates, namely the real numbers $x^1, x^2, \ldots$ such as the positions and momenta that one would like to use to parameterize points of the phase space, actually do not commute. This implies that the above classical geometrical framework is too narrow to describe in a faithful manner the physical spaces of great interest that prevail when one deals with microscopic systems.

This entices one to extend the duality

$$\underline{\text{Geometric Space}} \quad | \quad \underline{\text{Commutative Algebra}} \tag{14}$$

which plays a central role in algebraic geometry.

The point of departure of noncommutative geometry is the existence of natural spaces playing an essential role both in mathematics and in physics but whose "algebra of coordinates" is no longer commutative. The first examples came from

[10] not more than Hilbert in his list of 23 problems at the turn of the century.

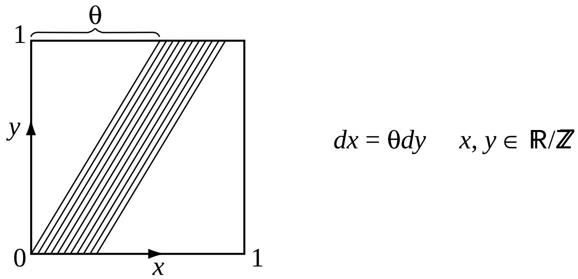

FIGURE 8. Foliation of the two-torus whose leaf space is $\mathbb{T}^2_\theta$

Heisenberg and the phase space in quantum mechanics but there are many others, such as the leaf spaces of foliations, duals of nonabelian discrete groups, the space of Penrose tilings, the noncommutative torus $\mathbb{T}^2_\theta$ which plays a role in the quantum Hall effect and in M-theory compactification [20] and the space of $\mathbb{Q}$-lattices [31] which is a natural geometric space, with an action of the scaling group providing a spectral interpretation of the zeros of the L-functions of number theory and an interpretation of the Riemann-Weil explicit formulas as a trace formula [19]. Another rich class of examples arises from deformation theory, such as deformation of Poisson manifolds, quantum groups and their homogeneous spaces. Moduli spaces also generate very interesting new examples as in [20] [50] as well as the fiber at ∞ in arithmetic geometry [34].

The common feature of many of these spaces is that, when one tries to analyze them from the usual set theoretic point of view, the usual tools break down for the following simple reason. Even though as a set they have the cardinality of the continuum, it is impossible to tell apart their points by a finite (or even countable) set of explicit functions. In other words, any *explicit* countable family of invariants fails to separate points and the *effective* cardinality is not the same as that of the continuum.

The general principle that allows one to construct the algebra of coordinates on such quotient spaces $X = Y/\sim$ is to replace the *commutative* algebra of functions on Y which are constant along the classes of the equivalence relation $\sim$ by the *noncommutative* convolution algebra of the equivalence relation so that the above duality gets extended as

(15) $\quad$ $\underline{\text{Geometric Quotient Space}}$ | $\underline{\text{Non Commutative Algebra}}$

The "simplest" non-trivial new spaces are the noncommutative tori which were fully analyzed at a very early stage of the theory in 1980 ([9]). Here $X = \mathbb{T}^2_\theta$ is the leaf space of the foliation $dx = \theta dy$ of the two torus $\mathbb{R}^2/\mathbb{Z}^2$ (*cf.* Fig. 8). If one tries to describe X in a classical manner by a commutative algebra of coordinates one finds that when θ is irrational, all "measurable functions" on X are almost everywhere constant and there are no non-constant continuous functions.

If one applies the above general principle one finds a very interesting algebra. This example played a crucial role as a starting point of the general theory thanks to the "integrality" phenomenon which was discovered in 1980 ([9]). Indeed even though the "shadow" of $\mathbb{T}^2_\theta$ obtained from the range of Morse functions can be a totally disconnected cantor set, and the dimension of the analogue of vector bundles is in general irrational, when one forms the "integral curvature" of these bundles as in the Gauss Bonnet theorem, one miraculously finds an integer.

This fact together with the explicit form of connections and curvature on vector bundles on $\mathbb{T}^2_\theta$ ([9]) were striking enough to suggest that ordinary differential geometry and the Chern-Weil theory could be successfully extended beyond their "classical" commutative realm.

A beginner might be tempted to be happy with the understanding of such simple examples as $\mathbb{T}^2_\theta$ ignoring the wild diversity of the general landscape. However the great variety of examples forces one to cope with the general case and to extend most of our geometric concepts to the general noncommutative case.

Usual geometry is just a particular case of this new theory, in the same way as Euclidean and non Euclidean geometry are particular cases of Riemannian geometry. Many of the familiar geometrical concepts do survive in the new theory, but they acquire also a new unexpected meaning.

Indeed even at the coarsest level of understanding of a space provided by measure theory, which in essence only cares about the "quantity of points" in a space, one finds unexpected completely new features in the noncommutative case. While it had been long known by operator algebraists that the theory of von-Neumann algebras represents a far reaching extension of measure theory, the main surprise which occurred at the beginning of the seventies [5] is that such an algebra M inherits from its noncommutativity a god-given time evolution:

$$\delta : \quad \mathbb{R} \quad \longrightarrow \quad \mathrm{Out}\, M \tag{16}$$

where $\mathrm{Out}\, M = \mathrm{Aut}\, M / \mathrm{Int}\, M$ is the quotient of the group of automorphisms of M by the normal subgroup of inner automorphisms. This led in my thesis [6] to the reduction from type III to type II and their automorphisms and eventually to the classification of injective factors.

The development of the topological ideas was made possible by Gelfand's C^*-algebras which he discovered early on in his mathematical life and was prompted by the Novikov conjecture on homotopy invariance of higher signatures of ordinary manifolds as well as by the Atiyah-Singer Index theorem. It has led to the recognition that not only the Atiyah-Hirzebruch K-theory but more importantly the dual K-homology as developed by Atiyah, Brown-Douglas-Fillmore and Kasparov admit noncommutative C^*-algebras as their natural framework. The cycles in K-homology are given by Fredholm representations of the C^*-algebra A of continuous functions. A basic example is the group C^*-algebra of a discrete group and restricting oneself to commutative algebras *i.e.* to commutative groups is an obviously undesirable assumption.

The development of differential geometric ideas, including de Rham homology, connections and curvature of vector bundles... took place during the eighties thanks to cyclic cohomology. It led for instance to the proof of the Novikov conjecture for hyperbolic groups but got many other applications. Basically, by extending the Chern-Weil characteristic classes to the general framework it allows for many concrete computations on noncommutative spaces.

The very notion of Noncommutative Geometry comes from the identification of the two basic concepts in Riemann's formulation of Geometry, namely those of manifold and of infinitesimal line element. It was recognized at the beginning of the eighties that the formalism of quantum mechanics gives a natural place not only to continuous variables of arbitrary dimension but also to infinitesimals (the compact operators in Hilbert space) and to the integral (the logarithmic divergence in an operator trace) as we shall explain below in section 3.3. It was also recognized long ago by geometors ([59], [53], [58]) that the main quality of the homotopy type of a manifold, (besides being defined by a cooking recipe) is to satisfy Poincaré duality not only in ordinary homology but in K-homology with the Fredholm module associated to the Dirac operator as the "fundamental class".

In the general framework of Noncommutative Geometry the confluence of the two notions of metric and fundamental class in K-homology for a manifold led very naturally to the equality

$$ds = 1/D, \tag{17}$$

which expresses the infinitesimal line element ds as the inverse of the Dirac operator D, hence under suitable boundary conditions as a propagator. The significance of D is two-fold. On the one hand it defines the line element by the above equation which allows one to compute distances by formula (18) below, on the other hand its homotopy class represents the K-homology fundamental class of the space under consideration.

It is worthwhile to explain in simple terms how noncommutative geometry modifies the measurement of distances. Such a simple description is possible because the evolution between the Riemannian way of measuring distances and the new (noncommutative) way exactly parallels the improvement of the standard of length[11] in the metric system. The original definition of the meter at the end of the 18th century was based on a small portion (one forty millionth part) of the size of the largest available macroscopic object (here the earth circumference). Moreover this "unit of length" became concretely represented in 1799 as "mètre des archives" by a platinum bar localized near Paris. The international prototype was a more stable copy of the "mètre des archives" which served to define the meter. The most drastic change in the definition of the meter occurred in 1960 when it was redefined as a multiple of the wavelength of a certain orange spectral line in the light emitted by isotope 86 of krypton. This definition was then replaced in 1983 by the current definition which using the speed of light as a conversion factor is expressed in terms of inverse-frequencies rather than wavelength, and is based on a hyperfine transition in the caesium atom. The advantages of the new standard are obvious. No comparison to a localized "mètre des archives" is necessary, the uncertainties are estimated as 10^{-15} and for most applications a commercial caesium beam is sufficiently accurate. Also we could (if any communication were possible) communicate our choice of unit of length to aliens, and uniformize length units in the galaxy without having to send out material copies of the "mètre des archives"!

The concept of "metric" in noncommutative geometry is precisely based on such a spectral data. Distances are no longer measured by (9) *i.e.* as the infimum of the

[11] or equivalently of time using the speed of light as a conversion factor.

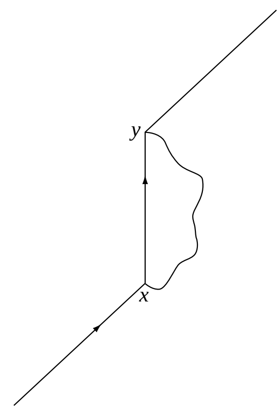

FIGURE 9. ds= Fermion Propagator

integral of the line element along arcs γ but as the supremum of the differences $|f(x) - f(y)|$ of scalar valued functions f subject to the constraint that they do not vary too rapidly as controlled by the operator norm of the commutator, $||[D, f]|| \leq 1$, so that

$$d(x, y) = \text{Sup}\,\{|f(x) - f(y)|\,;\, ||[D, f]|| \leq 1\} \tag{18}$$

This allows for a far reaching extension of the notion of Riemannian manifold given in spectral terms as an irreducible representation in Hilbert space $\mathcal{H}$ not only of the algebra $\mathcal{A}$ of coordinates on the geometric space but also of the line element $ds = D^{-1}$. Thus a noncommutative geometry is described as a "spectral triple"

$$(\mathcal{A}, \mathcal{H}, D)$$

The obtained paradigm of geometric space is very versatile and adapts to the following situations

- Leaf spaces of foliations
- Infinite dimensional spaces (as in supersymmetry)
- Fractal geometry
- Flag manifolds in quantum group theory[12]
- Brillouin zone in quantum physics
- Space-time

It also makes it possible to incorporate the "quantum corrections" of the geometry of space-time from the dressing of the line element ds identified with the propagator of fermions.

[12] *cf.* [47].

2.7. **Grothendieck's motives.**

The paradigm for what is a *space* in algebraic geometry has evolved considerably in the second half of the XXth century under the impulsion of specific problems such as the Weil conjectures which were finally proved by Deligne in 1973.

After the fundamental work of Serre who developed the theory of coherent sheaves and a very flexible notion of algebraic variety, based on Leray's notion of sheaves and Cartan's of "ringed space", Grothendieck undertook the task of extending the whole theory to the framework of *schemes* obtained by patching together the geometric counterpart of arbitrary commutative rings.

The "standard conjectures" of Grothendieck are a vast generalization of the Weil conjectures to
arbitrary correspondences, whereas the Weil case is confined to a specific correspondence known as the "Frobenius" correspondence. If true (together with conjectures of Hodge and Tate) the standard conjectures would allow for the construction of an abelian category of "motives" which unifies etale ℓ-adic cohomology for different values of ℓ with de Rham and Betti cohomologies.

So far only the "derived category" $DM(\mathcal{S})$ of the category of motives has been successfully constructed by Levine and Voevodski. In the long run, one of the essential objects of study in the theory are the L-functions associated to the m-th cohomology H^m of an algebraic variety defined over a number field. Their definition involves various cohomology theories which are only "unified" by the putative theory of motives. Moreover their properties including holomorphy are still conjectural and are a key motivation in the Langlands's program.

One might at first sight think that the theory of motives is of a totally different nature than the "analytic" objects involved in noncommutative geometry. This impression is quickly dispelled if one is aware of the Langlands's correspondence where the automorphic representations occurring on the analytic side appear as potential realizations of the "motives". In the last section of these notes we shall explain how the theory of motives appears (from motivic Galois theory) in the theory of renormalization in our joint work with M. Marcolli [33].

There is in fact an intriguing analogy between the motivic constructions and those of KK-theory and cyclic cohomology in noncommutative geometry.

Indeed the basic steps in the construction of the category $DM(\mathcal{S})$ of Voevodsky which is the "derived category" of the sought for category of motives are parallel to the basic steps in the construction of the Kasparov bivariant theory KK. The basic ingredients are the same, namely the correspondences which, in both cases, have a finiteness property "on one side". One then passes in both cases to complexes which in the case of KK is achieved by simply taking formal finite differences of "infinite" correspondences. Moreover, the basic equivalence relations between these "cycles" includes homotopy in very much the same way as in the theory of motives. Also as in the theory of motives one obtains an additive category which can be viewed as a "linearization" of the category of algebras. Finally one should note in the case of KK, that a slight improvement (concerning exactness) and a great technical simplification are obtained if one considers "deformations" rather than correspondences as the basic "cycles" of the theory, as is achieved in the E-theory.

Next, when instead of working over $\mathbb{Z}$ one considers the category $DM(k)_{\mathbb{Q}}$ obtained by tensorization by $\mathbb{Q}$, one can pursue the analogy much further and make contact with cyclic cohomology, where also one works rationally, with a similar role of filtrations. There also the obtained "linearization" of the category of algebras is fairly explicit and simple in noncommutative geometry. The obtained category is just the category of Λ-modules, based on the cyclic category Λ which will be described below in section 3.5.

2.8. **Topos theory.**

As mentioned above the notion of *scheme* is obtained by patching together the geometric counterpart of arbitrary commutative rings. Thus one might wonder at first why such patching is unnecessary in noncommutative geometry whose basic data is simply that of a noncommutative algebra. The main point there is that the noncommutativity present already in matrices allows one to perform this patching without exiting from the category of algebras.

Thus, exactly as above when defining the algebra of coordinates on a quotient as the convolution algebra of the equivalence relation, one implements the patching in an algebraic manner from the convolution algebra of a groupoid which is specified by the geometric recipe. In the case of a projective variety $X \subset \mathbb{P}_n(\mathbb{C})$ for instance, the Karoubi-Jouanolou trick consists in writing X as the quotient of an affine subvariety of the affine variety of idempotents $e^2 = e$, $e \in M_{n+1}(\mathbb{C})$.

The key conceptual notion which allows one to compare the two ways of proceeding in simple "affine" cases is the notion of Morita equivalence due to M. Rieffel [54] in the framework of C^*-algebras.

Thus there is no need in noncommutative geometry to give a "gluing data" for a bunch of commutative algebras, instead one sticks to the "purest" algebraic objects by allowing simply noncommutative algebras on the algebraic side of the basic duality,

$$\text{(19)} \qquad \underline{\text{Geometric Space}} \quad | \quad \underline{\text{Non Commutative algebra}}$$

It would be wrong however to think that algebraic geometry has no room for the new type of spaces associated to groupoids or simply groups. Indeed as an outgrowth of his construction of etale cohomology Grothendieck developed the general theory of sites[13] which generalize the notion of topological space, replacing the partially ordered set of open sets by a category in which the notion of "open cover" is assumed as an extra data. He then went much further by abstracting under the name of "topos" the properties of the category of sheaves of sets over a site, thus obtaining a vast generalization of topological spaces. This theory was used successfully in logics where it allows for a simple exposition of Cohen's independence result.

There are some similarities between noncommutative geometry and the theory of topoi as suggested by the diagram proposed by Cartier in [3], in the case of the cross product of a space by a group action or of a foliation which can be "treated" in the two ways,

[13] also called Grothendieck topologies.

$$
\begin{array}{ccccc}
 & & & & \text{Manifold } X \text{ with} \\
G\text{-space X} & & & & \text{foliation } \mathcal{F} \\
 & \searrow & & \swarrow & \\
 & & \text{Topological Groupoïd } \mathcal{G} & & \\
\text{Algebraic Geometry} \downarrow & & & & \downarrow \text{Noncommutative Geometry} \\
\text{Topos of } \mathcal{G}\text{-sets} & & \overset{?}{\longleftarrow \cdots\cdots\cdots \longrightarrow} & & C^*\text{-algebra } C^*(\mathcal{G})
\end{array}
$$

It is crucial to understand that the algebra associated to a topos does not in general allow one to recover the topos itself in the general noncommutative case.

To see the simplest example where two different topos give the same algebra it is enough to compare the topos $\mathcal{T}(X,.)$ of sheaves on a finite set X with the topos $\mathcal{T}(.,G)$ of G-sets where G is a finite group. If G is abelian of the same cardinality as X the algebra $C^*(G)$ associated to G is isomorphic thanks to the Fourier transform to $C(X)$ while the two corresponding topos are not isomorphic, since the set of "points" of $\mathcal{T}(X,.)$ is identified with X while the topos $\mathcal{T}(.,G)$ has only one point[14].

It then becomes clear that the invariants that are defined directly in terms of the algebra possess remarkable "stability" properties which would not be apparent in the topos side. To take an example the signature of a non-simply connected manifold M, when viewed as an element in the K-group of the group C^*-algebra $C^*(\pi_1(M))$ of the fundamental group $\pi_1(M)$ of M is a homotopy invariant. But its counterpart on the "topos" side which is the Novikov higher signature is not known to be invariant. In a similar manner the tools of analysis such as "positivity" (*cf.* section 3.1) can only be brought to bear in the right hand column of the above diagram as becomes apparent for instance with the vanishing of the $\hat{A}$-genus of manifolds admitting foliations with leaves of positive scalar curvature (*cf.* [13]).

In the context of algebraic geometry the basic duality (19) should of course not be restricted to "involutive" algebras over $\mathbb{C}$, and one should allow algebras with non trivial nilradical as one does in the commutative framework. In fact the gluing procedure naturally involves triangular matrices in that context. It is remarkable though that, except for "positivity", most of the tools that have been developed in the context of operator algebras actually apply in this broader framework of general algebras.

3. Fundamental Tools

I remember a discussion in the cafeteria of IHES several years ago with a group of mathematicians. We were discussing the tools that we currently use in doing mathematics and to make things simple each of us was only allowed to mention one, with of course the requirement that it should be simple enough.

There is no point in trying to give an exhaustive list, what I shall do rather is to illustrate a few examples of these tools taking noncommutative geometry as a pretext.

---

[14] *cf.* [49] exercise VII 2.

3.1. **Positivity.**

It is the key ingredient of measure theory. Probabilities are real numbers in the interval $[0,1]$ and there is no way to fool around with that.

Quantum theory tells us that the reason for positivity of probabilities is the presence of complex *probability amplitudes* and that in the quantum world these amplitudes behave additively while in the classical world it is the probabilities which behave additively.

The following inequality is in fact the corner stone of the theory of operator algebras

$$Z^* Z \geq 0 \,, \quad \forall Z \in A \tag{20}$$

Gelfand's C^*-algebras are those abstract algebras over $\mathbb{C}$ endowed with an antilinear involution $Z \to Z^*$, for which the above inequality "makes sense" *i.e.* defines a cone $A^+ \subset A$ of positive elements which possesses the expected properties. Thanks to functional analysis the whole industry of the theory of convexity can then be applied: one uses the Hahn Banach theorem to get positive linear forms, one constructs Hilbert spaces from positive linear forms, and all the powerful properties of operators in Hilbert space can then be used in this seemingly abstract context.

Likewise positivity plays a key role in physics under the name of unitarity which rules out any physical theory in which computed probabilities do not fulfill the golden rule

$$P(X) \in [0,1] \tag{21}$$

3.2. **Cohomology.**

To understand what cohomology is about one should start from a simple question and feel the need for an abstract tool. As a simple example we'll start from the Jordan curve theorem, which states that the complement in $\mathbb{P}_1(\mathbb{C})$ of a continuous simple closed curve admits exactly two connected components. One can try proving this with rudimentary tools but one should be aware of the existence of a Jordan curve C whose two dimensional Lebesgue measure is positive. This shows that the generic intersection $C \cap L$ of C with a line L will have positive one dimensional length and prevents one from giving a too naive argument involving *e.g.* the parity of the number of elements of $C \cap L$.

Cohomology theories such as K-theory for instance give an easy proof. For a compact space X the Atiyah-Hirzebruch K-theory $K(X)$ is obtained as the Grothendieck group of stable isomorphism classes of vector bundles. The main result is the Bott periodicity which gives a six term exact sequence corresponding to any closed subset $Y \subset X$. The point though is that while $K^j(X)$ and $K^j(Y)$ give four of these six terms the other two only depend upon the open set $X \backslash Y$ (or if one wants of its one point compactification). This *excision* allows one to easily handle delicate situations such as the one provided by the above Jordan curve C.

The Atiyah-Hirzebruch K-theory $K(X)$ is in fact the K-group of the associated C^*-algebra $C(X)$ of continuous functions on X and the theory keeps its essential features such as Bott periodicity and the six term exact sequence on the category of all C^*-algebras.

It is however vital to develop in this general framework the analogue of the Chern-Weil theory of curvature and characteristic classes of vector bundles. The main point is to be able to do computations of differential geometric nature in the above framework where *analysis* occupies the central place. It is worth quoting what Grothendieck says [41] in comparing the landscape of analysis in which he first worked with that of algebraic geometry in which he spent the rest of his mathematical life:

"Je me rappelle encore de cette impression saisissante (toute subjective certes), comme si je quittais des steppes arides et revêches, pour me retrouver soudain dans une sorte de "pays promis" aux richesses luxuriantes, se multipliant à l'infini partout où il plait à la main de se poser, pour cueillir ou pour fouiller...."

It is quite true that the framework of analysis is a lot less generous at first sight but my feeling at the end of the seventies was that the main reason for that was the absence of appropriate flexible tools similar to those provided by the calculus which would allow one to gradually develop an understanding by performing relatively easy and meaningful computations.

What was usually happening was that one would be confronted with problems that would just be too hard to cope with, so that the choice was between giving up or spending a considerable amount of time in trying to solve technical questions that would have meaning only to real specialists.

The situation would be totally different if, as was the case in differential geometry, one would have flexible tools such as de Rham currents and forms, allowing one to get familiar with simple examples and see one's way.

This is the main "philosophical" reason why I undertook in 1981 to develop cyclic cohomology which plays exactly that role in noncommutative geometry.

3.3. **Calculus.**

The infinitesimal calculus is built on the tension expressed in the basic formula

$$\int_a^b df \;=\; f(b) - f(a) \tag{22}$$

between the integral and the infinitesimal variation df of a function f. One gets to terms with this tension by developing the Lebesgue integral and the notion of differential form. At the intuitive level, the naive picture of the "infinitesimal variation" df as the increment of f for very nearby values of the variable is good enough for most purposes, so that there is no need for trying to create a theory of infinitesimals.

The scenery is different in noncommutative geometry, where quantum mechanics provides from the start a natural stage in which the notion of *variable* acquires a new and suggestive meaning.

In the classical formulation a real variable is seen as a map f from a set X to the real line. There is of course a large amount of arbitrariness in X that does not really affect the variable f *i.e.* for instance does not alter its range $f(X)$ (the subset of $\mathbb{R}$ of values that are reached). In quantum mechanics there is a stage which is fixed once and for all, it is a separable Hilbert space $\mathcal{H}$. Note that all infinite dimensional Hilbert spaces are isomorphic, so that this stage is fairly *canonical*. What people have found in developing quantum mechanics is that instead of dealing with real variables which are just maps f from a set X to the real line one has to replace that notion by that of *self-adjoint* operator in the Hilbert space $\mathcal{H}$.

These "new" variables share many features with the classical ones, for instance the role of the range $f(X)$ is now played by the *spectrum* of the self-adjoint operator. This spectrum is also a subset of $\mathbb{R}$ and some of its points can be reached more often than others. The number of time an element of the spectrum is reached is known as the spectral multiplicity.

Another rather amazing compatibility of the new notion with the old one is that one can compose any (measurable) function $h : \mathbb{R} \to \mathbb{R}$ with a real variable in the quantum sense. That amounts to the borel functional calculus for self-adjoint operators. Given such a T not only does $p(T)$ make sense in an obvious way when p is a polynomial, but in fact this definition extends by continuity to all borel functions !!

Once one has acquired some familiarity with the new notion of variable in quantum mechanics one easily realizes that it is a perfect home for infinitesimals, namely for variables that are smaller than ϵ for any ϵ, without being zero. Of course, requiring that the operator norm is smaller than ϵ for any ϵ is too strong, but one can be more subtle and ask that, for any ϵ positive, one can condition the operator by a finite number of linear conditions, so that its norm becomes less than ϵ. This is a well known characterization of compact operators in Hilbert space, and they are the obvious candidates for infinitesimals.

The basic rules of infinitesimals are easy to check, for instance the sum of two compact operators is compact, the product compact times bounded is compact and they form a two sided ideal $\mathcal{K}$ in the algebra of bounded operators in $\mathcal{H}$.

The size of an infinitesimal $\boldsymbol{\epsilon} \in \mathcal{K}$ is governed by the rate of decay of the decreasing sequence of its characteristic values $\mu_n = \mu_n(\boldsymbol{\epsilon})$ as $n \to \infty$. (By definition $\mu_n(\boldsymbol{\epsilon})$ is the n'th eigenvalue of the absolute value $|\boldsymbol{\epsilon}| = \sqrt{\boldsymbol{\epsilon}^* \boldsymbol{\epsilon}}$). In particular, for all real positive α, the following condition defines infinitesimals $\boldsymbol{\epsilon}$ of order α:

$$\mu_n(\boldsymbol{\epsilon}) = O(n^{-\alpha}) \qquad \text{when } n \to \infty. \tag{23}$$

Infinitesimals of order α also form a two–sided ideal and, moreover,

$$\boldsymbol{\epsilon_j} \text{ of order } \alpha_j \Rightarrow \boldsymbol{\epsilon_1 \epsilon_2} \text{ of order } \alpha_1 + \alpha_2 \,. \tag{24}$$

This is just a translation of well known properties of compact operators in Hilbert space but the really new ingredient in the calculus of infinitesimals in noncommutative geometry is the integral.

$$⨍ \boldsymbol{\epsilon} \in \mathbb{C} \,. \tag{25}$$

where $\boldsymbol{\epsilon}$ is an infinitesimal of order one. Its construction [15] rests on the analysis of the logarithmic divergence of the ordinary trace for an infinitesimal of order one, mainly due to Dixmier ([38]). This trace has the usual properties of additivity and positivity of the ordinary integral, but it allows one to recover the power of the usual infinitesimal calculus, by automatically neglecting the ideal of infinitesimals of order > 1

$$⨍ \boldsymbol{\epsilon} \, = \, 0 \,, \quad \forall \boldsymbol{\epsilon} \,, \quad \mu_n(\boldsymbol{\epsilon}) = o(n^{-1}) \,. \tag{26}$$

By filtering out these operators, one passes from the original stage of the quantized calculus described above to a *classical* stage where the notion of locality finds its correct place.

Using (26) one recovers the above mentioned tension of the ordinary differential calculus, which allows one to neglect infinitesimals of higher order (such as $(df)^2$) in an integral expression.

3.4. **Trace and Index Formulas.**

The Atiyah-Singer index formula is an essential motivation and ingredient in noncommutative geometry.

First the flexibility that one gains in the noncommutative case by considering groupoids and not just spaces allows for a very simple geometric formulation of the index theorem of Atiyah-Singer, in which the "analysis" is subsumed by a geometric picture. The obtained geometric object, called the "tangent groupoid" $TG(M)$ of a manifold M, is obtained by gluing the tangent bundle $T(M)$ (viewed as the groupoid which is the union of the additive groups given by the tangent spaces $T_x(M)$) with the space $M\times M\times]0,1]$, viewed as the union of the "trivial" groupoids $M\times M$ where $(x,y)\circ(y,z) = (x,z)$. To the inclusion $T(M)\subset TG(M)$ corresponds an exact sequence of the corresponding groupoid C^*-algebras and one finds without effort that the connecting map for the six term K-theory exact sequence is exactly the analytical index in the Atiyah-Singer index formula.

The proof of the K-theory formulation of the Atiyah-Singer index formula then follows from the analogue of the Thom isomorphism in the noncommutative case *cf.* [15].

The power of the Atiyah-Singer index formula is however greatly enhanced when it is formulated not in K-theory but when the Chern character is used to express the topological side of the formula in "local" terms using characteristic classes. The first decisive step in that direction had been taken in the special case of the signature operator by Hirzebruch based on Thom's cobordism theory. In classical differential geometry the Chern-Weil theory was available before the index formula and greatly facilitated the translation from K-theory to the ordinary cohomological language.

In noncommutative geometry the analogue of the Chern-Weil theory, namely cyclic cohomology had to be developed first as a necessary preliminary tool towards the analogue of the "local" Atiyah-Singer index formula.

The general form of the local index formula in noncommutative geometry was obtained by H. Moscovici and myself in 1996, [25]. The basic conceptual notion that emerges is that the notion of locality is recovered in the absence of the usual "point set" picture by passing to the Fourier dual *i.e.* by expressing everything in "momentum space" where what was local *i.e.* occurring in the small in coordinate space now occurs "in the large" *i.e.* as asymptotics. Moreover the "local expressions" are exactly those which are obtained from the noncommutative integral given by the Dixmier trace discussed above, suitably extended (as first done by Wodzicki in the case of classical pseudo-differential operators [60]) to allow for the integration of infinitesimals of order smaller than one[15]. The notion of curvature in noncommutative geometry is precisely based on these formulas.

Moreover while leaf spaces of foliations were the motivating example for this theorem, another spin off was the development of the cyclic cohomology for Hopf algebras which will be briefly discussed below.

The Lefschetz formulas which in particular imply fixed point results are of the same nature as the index formula. In the simplest instance they are obtained by comparing the results in computing the trace of an operator in two different manners. One is "analytic" and consists in adding the eigenvalues of the operator. The other is "geometric" and is obtained by adding up the diagonal elements of the matrix of the operator or in general by integrating the Schwartz kernel of the operator along the diagonal. The Selberg trace formula and its various avatars play a crucial role in the Langlands's program, which has been successfully advanced in the work of Drinfeld and Lafforgue in the case of function fields.

An n-dimensional $\mathbb{Q}$-lattice ([31]) consists of an ordinary lattice Λ in $\mathbb{R}^n$ and a homomorphism

$$\phi : \mathbb{Q}^n/\mathbb{Z}^n \to \mathbb{Q}\Lambda/\Lambda.$$

Two such $\mathbb{Q}$-lattices are *commensurable* if and only if the corresponding lattices are commensurable and the maps agree modulo the sum of the lattices. The noncommutative space $\mathcal{L}_n$ of $\mathbb{Q}$-lattices ([31]) is obtained as the quotient by the relation of *commensurability*. The group $\mathbb{R}_+^*$ acts on $\mathcal{L}_n$ by scaling. It is quite remarkable that that the zeros of the Riemann zeta function appear as an absorption spectrum (*cf.* Fig. 7) of the scaling action in the L^2 space of the space of commensurability classes of one dimensional $\mathbb{Q}$-lattices as in [19]. In fact the Galois group G of $\mathbb{Q}^{ab}$ acts on $\mathcal{L}_1$ in a natural manner and the above L^2 space splits as a direct sum labelled by characters of G with spectra of the corresponding L-functions appearing in each of the sectors. Moreover the Riemann-Weil explicit formulas appear from a trace formula ([19], [52]) deeply related to the validity of RH.

In the case of the L-functions associated to an arithmetic variety, the search for a unified form of the local factors has lead Deninger [37] to hope for the construction of an hypothetical "arithmetic site" whose expected properties are very reminiscent of the space of $\mathbb{Q}$-lattices. Since that latter space does provide a simple explanation as a trace formula of Lefschetz type for the local factors of Hecke L-functions it is natural to extend it to cover the case of L-functions associated to an arithmetic variety. A first step in that direction is done in [32] and hinges on the confluence between the theory of motives which underlies the cohomologies involved in the construction of the L-functions and noncommutative geometry which underlies the analysis of spaces such as the space of $\mathbb{Q}$-lattices.

3.5. **Abelian categories.**

The language of categories is omnipresent in modern algebraic geometry. Homological algebra is a tool of great power which is available as soon as one deals with abelian categories. Moreover the category of finite dimensional representations of an affine group scheme can be characterized abstractly as a "tannakian" category *i.e.* a tensor category fulfilling certain natural properties. It is striking that this

[15] *cf.* formula (39) of section 4.1.

result was obtained by the physicists Doplicher and Roberts in their work on superselection sectors in algebraic quantum field theory and independently by Deligne in the context of algebraic geometry.

Exactly as what happens for schemes, the category of noncommutative algebras is not even an additive category since the sum of two algebra homomorphisms is in general not a homomorphism.

In order to be able to use the arsenal of homological algebra one embedds the above category in an abelian category, the category of Λ-modules, using the cyclic category Λ [12].

The resulting functor

$$A \to A^{\natural}$$

should be compared to the embedding of a manifold in linear space. It allows one to treat algebras as objects in an abelian category for which many tools such as the bifunctors

$$\mathrm{Ext}^n(X, Y)$$

are readily available. The key ingredient is the *cyclic category*. It is a small category which has the same classifying space as the compact group $U(1)$.

It can be defined by generators and relations. It has the same objects as the small category Δ of totally ordered finite sets and increasing maps which plays a key role in simplicial topology. Let us recall that Δ has one object $[n]$ for each integer n, and is generated by faces $\delta_i, [n-1] \to [n]$ (the injection that misses i), and degeneracies $\sigma_j, [n+1] \to [n]$ (the surjection which identifies j with $j+1$), with elementary relations. To obtain the cyclic category Λ one adds for each n a new morphism $\tau_n, [n] \to [n]$ such that,

$$\begin{gathered} \tau_n \delta_i = \delta_{i-1} \tau_{n-1} \quad 1 \leq i \leq n\,, \quad \tau_n \delta_0 = \delta_n \\ \tau_n \sigma_i = \sigma_{i-1} \tau_{n+1} \quad 1 \leq i \leq n\,, \quad \tau_n \sigma_0 = \sigma_n \tau_{n+1}^2 \\ \tau_n^{n+1} = 1_n\,. \end{gathered}$$

The original definition of Λ (cf. [15]) used homotopy classes of non decreasing maps from S^1 to S^1 of degree 1, mapping $\mathbb{Z}/n$ to $\mathbb{Z}/m$ and is trivially equivalent to the above.

Given an algebra A one obtains a module $A^{\natural}$ over the small category Λ by assigning to each integer $n \geq 0$ the vector space $C^n = A^{\otimes n+1}$ while the basic operations are given by

$$\begin{gathered} \delta_i(x^0 \otimes \ldots \otimes x^n) = x^0 \otimes \ldots \otimes x^i x^{i+1} \otimes \ldots \otimes x^n\,, \quad 0 \leq i \leq n-1 \\ \delta_n(x^0 \otimes \ldots \otimes x^n) = x^n x^0 \otimes x^1 \otimes \ldots \otimes x^{n-1} \\ \sigma_j(x^0 \otimes \ldots \otimes x^n) = x^0 \otimes \ldots \otimes x^j \otimes 1 \otimes x^{j+1} \otimes \ldots \otimes x^n, \quad 0 \leq j \leq n \\ \tau_n(x^0 \otimes \ldots \otimes x^n) = x^n \otimes x^0 \otimes \ldots \otimes x^{n-1}\,. \end{gathered}$$

These operations satisfy the contravariant form of the above relations. This shows that any algebra A gives rise canonically to a Λ-module[16] $A^\natural$ and gives a natural covariant functor from the category of algebras to the abelian category of Λ-modules. This gives [15] an interpretation of the cyclic cohomology groups $HC^n(A)$ as Ext^n functors, so that

$$HC^n(A) = \mathrm{Ext}^n(A^\natural, \mathbb{C}^\natural)\,.$$

All of the general properties of cyclic cohomology such as the long exact sequence relating it to Hochschild cohomology are shared by Ext of general Λ-modules and can be attributed to the equality of the classifying space $B\Lambda$ of the small category Λ with the classifying space BS^1 of the compact one-dimensional Lie group S^1. One has [12],

$$B\Lambda = BS^1 = P_\infty(\mathbb{C}) \tag{27}$$

3.6. **Symmetries.**

It is hard to overemphasize the power of the idea of symmetry in mathematics. In many cases it allows one to bypass complicated computations by guessing the answer from its invariance properties. It is precisely in the ability to bypass the computations that lies the power of modern mathematics inaugurated by the works of Abel and Galois. Let us listen to Galois:

"Sauter à pieds joints sur ces calculs, grouper les opérations, les classer suivant leurs difficultés et non leurs formes, telle est suivant moi, la mission des géomètres futurs."

Abel and Galois analyzed the symmetries of functions of roots of polynomial equations and Galois found that a function of the roots is a "rational" expression iff it is invariant under a specific group G of permutations naturally associated to the equation and to the notion of what is considered as being "rational". Such a notion defines a field K containing the field $\mathbb{Q}$ of rational numbers always present in characteristic zero. He first exhibits a rational function[17]

$$V(a, b, ..., z)$$

of the n distinct roots $(a, b, ..., z)$ of a given equation of degree n, which affects $n!$ different values under all permutations of the roots, *i.e.* which "maximally" breaks the symmetry. He then shows that there are n "rational" functions $\alpha(V), \beta(V), ...$ of V which give back the roots $(a, b, ..., z)$. His group G is obtained by decomposing in irreducible factors over the field K the polynomial (over K) of degree $n!$ of which V is a root. Using the above rational functions

$$\alpha(V_j), \beta(V_j), ...$$

applied to the other roots V_j of the irreducible factor which admits V as a root, yields the desired group of permutations[18] of the n roots $(a, b, ..., z)$.

This procedure has all the characteristics of **fundamental** mathematics:

- It bypasses complicated computations.

[16] The small category Λ being canonically isomorphic to its opposite there is no real difference between covariant and contravariant functors.

[17] which one can take as a linear form with coefficients in $\mathbb{Q}$.

[18] the root $V_1 = V$ gives the identity permutation.

- It focusses on the key property of the solution.
- It has bewildering power.
- It creates a new concept.

In noncommutative geometry the symmetries are encoded by Hopf algebras which are not required to be either commutative or cocommutative. Besides quantum groups which play in noncommutative geometry a role analogous to that of Lie groups in classical differential geometry, certain natural infinite dimensional Hopf algebras neither commutative nor cocommutative appeared naturally in the transverse geometry of foliations. At first it was discovered in the early eighties that the Godbillon-Vey class or more general Gelfand-Fuchs classes actually appear in the cyclic cohomology of the foliation algebras and allow one to relate purely differential geometric hypotheses with the finest invariants of the von-Neumann algebra of the foliation such as its flow of weights [13].

The construction of a spectral triple associated to the transverse geometry of an arbitrary foliation took a long time and was achieved in [25]. The elaborated computation of the local index formula for such a spectral triple required developing first the analogue of Lie algebra cohomology in the context of Hopf algebras which are not required to be either commutative or cocommutative. This was done in [26] and the corresponding cyclic cohomology of Hopf algebras plays in general the same role as the classical theory of characteritic classes for Lie groups. It allowed us to express the local index formula for transversal geometry of foliations in terms of Gelfand-Fuchs classes. Moreover it transits through the above category of Λ-modules which seems to play a rather "universal" role in cohomological constructions. The theory was extended recently to the much wider framework of anti Yetter-Drinfeld modules [42].

4. The input from Quantum Field Theory

The depth of mathematical concepts that come directly from physics has been qualified in the following terms by Hadamard :

"Not this short lived novelty which can too often influence the mathematician left to his own devices, but this infinitely fecund novelty which springs from the nature of things"

It is indeed quite difficult for a mathematician not to be attracted by the apparent mystery underlying renormalization, a combinatorial technique devised by hardcore physicists starting from 1947, to get rid of the undesirable "divergences" that plagued the computations of quantum field theory when they tried to go beyond the "tree level" approximation that was so succesfull in Dirac's hands with his computation of Einstein's A and B coefficients for the interaction of matter with radiation.

As is well known the renormalization technique when combined with the Standard model Lagrangian of physics is so successful that it predicts results (such as the anomalous moment of the electron) with bewildering precision[19]. Thus clearly there is "*something right*" there and it is crucial to try to understand what.

[19]the width of a hair in the distance Paris-New York.

As we shall explain below noncommutative geometry has its say both on the standard model of particle physics and on renormalization. In the latter case this comes from my joint works both with D. Kreimer [22] and with M. Marcolli [33].

4.1. **The Standard Model.**

One clear lesson of general relativity is that "gravity" is encoded by space-time geometry, while curvature plays a basic role through the action functional (13). Gravity is not the only "fundamental force" and the three others (weak, electromagnetic and strong) combine in an additional "matter" action corresponding to the five type of terms in the standard model Lagrangian. Thus the full action is of the form

$$S = S_E + S_G + S_{GH} + S_H + S_{Gf} + S_{Hf} \tag{28}$$

where S_E is the Einstein-Hilbert action (13), S_G is the Yang-Mills self-interaction of gauge bosons, S_H is the quartic self-interaction of higgs bosons, S_{GH} is the minimal coupling of gauge bosons with higgs bosons, etc...

The Minkowski geometry of space-time was deduced from the Maxwell part of the Lagrangian of physics and our aim is to incorporate in the model of space-time geometry the modifications which correspond to the additional terms of the weak and strong forces. We shall not address the important problem of the relation between the Euclidean and Minkowski frameworks and work only in the Euclidean signature.

Our starting point is the action functional (28) which we view as the best approximation to "physics up to TEV". We can start understanding something by looking at the symmetry group of this functional. If we were dealing with pure gravity, i.e. the Einstein theory alone, the symmetry group of the functional would just be the diffeomorphism group of the usual space-time manifold. But because of the contribution of the standard model the gauge theories introduce another large symmetry group namely the group of maps[20] from the manifold to the small gauge group, which as far as we know in the domain of energies up to a few hundred GEV, is $U_1 \times SU_2 \times SU_3$. The symmetry group G of the full functional S (28) is not the product of the diffeomorphism group by the group of gauge transformations of second kind, but is their semi-direct product. Gauge transformations and diffeomorphisms mix in the same way as translations and Lorentz transformations come together in the Poincaré group.

At this point, we can ask two very simple questions:

- Is there a space X such that $\mathrm{Diff}(X)$ coincides with the symmetry group G ?
- Is there a simple action functional that reproduces the action functional (28) when applied to X.

In other words we are asking to completely geometrize the effective model of space-time as pure gravity of the space X. Of course we don't believe that the standard model coupled to gravity is the "final word" but we think it is crucial to stay in pure

[20] called gauge transformations of second kind.

geometry even at the "effective" level so that we get a better idea of the structure of space-time based on experimental evidence.

Now, if we look for X among ordinary manifolds, we have no chance to find a solution since by a result of Mather and Thurston the diffeomorphism group of a (connected) manifold is a simple group. A simple group has no nontrivial normal subgroup, and cannot be a semi-direct product in a non-trivial way.

This obstruction disappears in the noncommutative world, where any variant of the automorphism group $\mathrm{Aut}^+(\mathcal{A})$ will automatically contain a non-trivial normal subgroup $\mathrm{Int}^+(\mathcal{A})$ of inner automorphisms, namely those of the form

$$\mathrm{Ad}_u\,(x) = \, u\,x\,u^*$$

It turns out that, modulo a careful discussion of the lifting of elements of $\mathrm{Aut}^+(\mathcal{A})$, there is one very natural non commutative algebra $\mathcal{A}$ whose symmetry is the above group G [56] [57]. The group of inner automorphisms corresponds to the group of gauge transformations and the quotient by inner corresponds to diffeomorphisms. It is comforting that the physics vocabulary is the same as the mathematical one. In physics one talks about *internal* symmetries and in mathematics about *inner* automorphisms.

The corresponding space is a product $X = M \times F$ of an ordinary manifold M by a *finite* noncommutative space F. The algebra $\mathcal{A}_F$ describing the finite space F is the direct sum of the algebras $\mathbb{C}$, $\mathbb{H}$ (the quaternions), and $M_3(\mathbb{C})$ of 3×3 complex matrices.

The standard model fermions and the Yukawa parameters (masses of fermions and mixing matrix of Kobayashi Maskawa) determine the spectral geometry of the finite space F in the following manner. The Hilbert space is finite-dimensional and admits the set of elementary fermions as a basis. For example for the first generation of quarks, this set is (with suitable "color" labels),

$$u_L, u_R, d_L, d_R, \bar{u}_L, \bar{u}_R, \bar{d}_L, \bar{d}_R\,. \tag{29}$$

The algebra $\mathcal{A}_F$ admits a natural representation in $\mathcal{H}_F$ (see [17]) and the Yukawa coupling matrix Y determines the operator D.

The detailed structure of Y (and in particular the fact that color is not broken) allows one to check the basic algebraic properties of noncommutative geometry. Among these an important one is the "order one" condition which means that for a suitable real structure J given by an antilinear involution on the Hilbert space $\mathcal{H}$ the following commutation relations hold,

$$[a, b^o] = 0 \quad [[D, a], b^o] = 0\,, \quad \forall a, b \in \mathcal{A}, \quad b^o = \, J b^* J^{-1}\,. \tag{30}$$

Now in the same way as diffeomorphisms of X admit inner perturbations, the metrics (given by the inverse line element D) admit inner perturbations. These come immediately from trying to transfer a given metric on an algebra $\mathcal{A}$ to a Morita equivalent algebra $\mathcal{B} = \mathrm{End}_{\mathcal{A}}(\mathcal{E})$ with $\mathcal{E}$ a finite projective right $\mathcal{A}$-module. Indeed while the Hilbert space $\mathcal{E} \otimes \mathcal{H}$ is easy to define using a Hermitian structure on $\mathcal{E}$, the extension of the operator D to $\mathcal{E} \otimes \mathcal{H}$ requires the choice of a "connection" *i.e.* of a linear map fulfilling the Leibniz rule,

$$\nabla \quad : \quad \mathcal{E} \to \mathcal{E} \otimes \Omega^1\,, \qquad \nabla(\xi\, a) = \, \nabla(\xi)\, a + \xi\, [D, a]\,, \tag{31}$$

where Ω^1 is the $\mathcal{A}$-bimodule of linear combinations of operators on $\mathcal{H}$ of the form $a\,[D,b]$ for $a,b \in \mathcal{A}$.

The extension of D to $\mathcal{E} \otimes \mathcal{H}$ is then simply given by

$$\tilde{D}(\xi \otimes \eta) = \nabla(\xi)\,\eta + \xi \otimes D\eta\,. \tag{32}$$

The "inner perturbations" of the metric then come from the obvious self-Morita equivalence of $\mathcal{A}$ with itself given by the right module $\mathcal{A}$, and those which preserve the real structure J are then of the form,

$$D \to D + A + JAJ^{-1}\,, \quad \forall A = A^* \in \Omega^1 \tag{33}$$

When one computes the inner perturbations of the product geometry $M \times F$ where M is a 4–dimensional Riemannian spin manifold one finds the standard model gauge bosons $\gamma, W^{\pm}, Z$, the eight gluons and the Higgs fields φ with accurate quantum numbers.

As it turns out the second of the above questions, namely finding a simple action principle that reproduces the action functional (28) also admits a remarkably nice answer. To understand it one first needs to reflect a bit on the notion of "observable" in gravity. The diffeomorphism invariance of the theory deprives the idea of a "specific point" of any intrinsic meaning and only quantities such as "diameter" etc.. that can be defined in a diffeomrphism invariant manner are "observable". It is easy to check that this is so then for all spectral properties of the Dirac operator. Thus a rather strong form of diffeomorphism invariance is obtained by imposing that the action functional is "spectral" *i.e.* entirely deduced from the spectrum of the operator D defining the metric.

The spectrum of D is analyzed by the counting function,

$$N(\Lambda) = \#\text{ eigenvalues of } D \text{ in } [-\Lambda, \Lambda]. \tag{34}$$

This step function $N(\Lambda)$ is the superposition of two terms,

$$N(\Lambda) = \langle N(\Lambda)\rangle + N_{\rm osc}(\Lambda). \tag{35}$$

where the "average part" is given as a sum labelled by the "dimension spectrum" S of the noncommutative space X under consideration as,

$$\langle N(\Lambda)\rangle = S_\Lambda(D) = \sum_{k\in S} \frac{\Lambda^k}{k} {\int\!\!\!\!\!\!-} |ds|^k + \zeta_D(0)\,, \tag{36}$$

where $\zeta_D(s) = \mathrm{Trace}(|D|^{-s})$ and the various terms ${\int\!\!\!\!\!\!-} |ds|^k$ are given as residues of $\zeta_D(s)$ at elements $k \in S$ of the dimension spectrum. The oscillatory part $N_{\rm osc}(\Lambda)$ is the same as for a random matrix and is not relevant here.

The detailed computation of the spectral action functional $S_\Lambda(D)$ in the case of the inner perturbations of the above space $X = M \times F$ is quite involved and we refer to ([4]) and ([46]) for the precise details. Both the Hilbert–Einstein action functional for the Riemannian metric, the Yang–Mills action for the vector potentials, the self interaction and the minimal coupling for the Higgs fields all appear with the correct signs to give the first four "bosonic" terms

$$S_{bos} = S_E + S_G + S_{GH} + S_H \tag{37}$$

while the last two "fermionic" terms in (28) are simply given from the start as

$$S_{fer} = S_{Gf} + S_{Hf} = \langle f, D\, f \rangle\,. \tag{38}$$

For instance to see why the Einstein-Hilbert action appears one can check the following computation of the two dimensional volume (*i.e.* the "area") of a four dimensional compact Riemannian manifold M_4 (cf. [45]),

$$⨍ ds^2 = -\frac{1}{24\,\pi^2} \int_{M_4} r\,\sqrt{g}\; d^4x \tag{39}$$

which should be compared with (13). Note that this result allows one to recover the full knowledge of the scalar curvature since one gets in fact for any function f on the manifold M_4 the equality,

$$⨍ f\, ds^2 = -\frac{1}{24\,\pi^2} \int_{M_4} r\, f\,\sqrt{g}\; d^4x \tag{40}$$

Two other terms appear besides $S_{bos} = S_E + S_G + S_{GH} + S_H$ in the computation of $\langle N(\Lambda)\rangle$ for the inner perturbations of the above space $X = M \times F$ they are

- The "cosmological" term $\Lambda^4 ⨍ ds^4$ where $⨍ ds^4$ is a universal constant times the riemannian volume.
- Weyl gravity terms involving the Weyl curvature and topological terms.

The next natural step is to try to make sense of a "Euclidean" functional integral of the form

$$\langle \sigma \rangle = \mathcal{N} \int \sigma(D, f) e^{-S_\Lambda(D) - \langle f,\, D\, f\rangle}\, D[D]\, D[f] \tag{41}$$

where $\mathcal{N}$ is a normalization factor, $\sigma(D, f)$ is a spectral observable, *i.e.* a unitarily covariant function of the self-adjoint operator D and f a vector $f \in \mathcal{H}$. The first difficulty is to write the constraint on the random hermitian operator D asserting that it is the inverse line element for a suitable geometry. It is there that the algebra $\mathcal{A}$ which is part of the spectral triple should enter the scene. Its role is to allow one to write an equation of cohomological nature defining the homology fundamental class of the geometric space *i.e.* a Hochschild 4-cocycle[21] $c = \sum a_0\, da_1 \cdots da_4$. The basic constraint of "Heisenberg" type is then

$$\sum a_0\, [D, a_1] \cdots [D, a_4] = \gamma_5\,. \tag{42}$$

By fixing the volume form (as the Hochschild class of c) it freezes the "Weyl" degree of freedom which is the only one to have a wrong sign in the Euclidean functional integral which is now only performed among "metrics" with a fixed volume form. Note that the a_j are "generators" of the algebra $\mathcal{A}$ whose only role is to provide the basic Hochschild 4-cocycle c so that the equation fulfilled by the a_j's viewed as operators in $\mathcal{H}$ is the vanishing of the Hochschild boundary of c. We began in [27], [28], [29] the investigation of the solutions of the above equations and of lower dimensional analogues.

[21] if we work in dimension 4.

4.2. **Renormalization.**

Of course the basic obstacle in dealing with functional integrals occuring in quantum field theory is that the expressions obtained from the perturbative expansion of the simplest functional integrals are usually divergent and require "renormalization". There is nothing "unphysical" with that, since the divergences come from the very nature of quantum field theory, and the removal of divergences comes naturally from the physics standpoint which teaches one to distinguish the experimentally measured quantities such as masses, charges etc... from the "bare" input in the mathematical equations.

The intricacies of the technique of renormalization were however sufficiently combinatorially involved to prevent an easy conceptual mathematical understanding. This state of affairs changed drastically in the recent years thanks to the following steps done in my collaboration with D. Kreimer:

- The discovery of a Hopf algebra underlying the BPHZ method of renormalization [48] [21].
- The analysis of the group[22] $\mathrm{Difg}(\mathcal{T})$ of diffeographisms of a given quantum field theory $\mathcal{T}$ [21].
- The discovery of the identity between the Birkhoff decomposition of loops in prounipotent Lie groups and the combinatorics of the minimal subtraction scheme in dimensional regularization [22].
- The construction for massless theories of an action of $\mathrm{Difg}(\mathcal{T})$ by formal diffeomorphisms of the coupling constants of the theory [22].

As a result of the above developments one can express the process of renormalization in the following conceptual terms, taking for $\mathcal{T}$ a massless theory with a single dimensionless coupling constant g. Let $g_{\mathrm{eff}}(z)$ be the unrenormalized effective coupling constant in dimension $D-z$, viewed as a formal power series in g and a function of the complex variable z. Let then

$$g_{\mathrm{eff}}(z) = g_{\mathrm{eff}_+}(z)\,(g_{\mathrm{eff}_-}(z))^{-1} \tag{43}$$

be the Birkhoff decomposition of this loop in the group of formal diffeomorphisms, with g_+ regular at $z=0$ and g_- regular outside $z=0$ and normalized to be 1 at ∞. Then the loop $g_{\mathrm{eff}_-}(z)$ is the bare coupling constant and $g_{\mathrm{eff}_+}(0)$ is the renormalized effective coupling.

4.3. **Symmetries.**

The Birkhoff decomposition (*i.e.* a decomposition such as (43)) plays a key role in dealing with the Riemann-Hilbert problem *i.e.* finding a differential equation with given singularities and given monodromies. Thus the role of the Birkhoff decomposition in the above conceptual understanding of the combinatorics of the subtraction procedure in renormalization suggested a potential relation between renormalization and the Riemann-Hilbert correspondence. The latter is a very broad theme in mathematics involving the classification of "geometric" datas such as differential equations and flat connections on vector bundles in terms of "representation theoretic" datas involving variations on the idea of monodromy. These

[22] and of its Lie algebra.

variations become more involved in the "irregular" singular case as in the work of Martinet-Ramis [51].

The Riemann-Hilbert correspondence underlying renormalization was unveiled in my joint work with M. Marcolli [33]. The "geometric" side in the correspondence is given by "equisingular" flat connections on vector bundles over a base space B which is described both in mathematical and in physics terms.

In "maths" terms B is the total space of a $\mathbb{G}_m$-principal bundle[23] over an infinitesimal disk Δ in $\mathbb{C}$. The connection is $\mathbb{G}_m$-invariant and singular on the "special fiber" over $0 \in \Delta$. The key "equisingularity" property is that the pull backs of the connection under sections of the bundle B which take the same value at $0 \in \Delta$ all have the same singularities at 0.

In "physics"terms the base Δ is the space of complexified dimensions around $D \in \mathbb{C}$ the dimension where one would like to do physics. The fibers of the $\mathbb{G}_m$-principal bundle correspond to normalization of the integral in complex dimensions as used by physicists in the dimensional regularization (Dim. Reg.) scheme. The physics input that the counterterms are independent of the additional choice of a unit of mass translates into the notion of equisingularity for the connection naturally provided by the computations of quantum field theory.

The "representation theoretic" side of our Riemann-Hilbert correspondence is defined first in an abstract and unambiguous manner as the affine group scheme which classifies the equisingular flat connections on finite dimensional vector bundles. What we show [33] is that when working with formal Laurent series over $\mathbb{Q}$, the data of equisingular flat vector bundles define a Tannakian category whose properties are reminiscent of a category of mixed Tate motives. This category is equivalent to the category of finite dimensional representations of a unique affine group scheme U^* and our main result is the explicit determination of the *" motivic Galois group"* U^*, which is uniquely determined and universal with respect to the set of physical theories. The renormalization group can be identified canonically with a one parameter subgroup of U^*.

As an algebraic group scheme, U^* is a semi-direct product by the multiplicative group $\mathbb{G}_m$ of a pro-unipotent group scheme whose Lie algebra is freely generated by one generator in each positive integer degree. In particular U^* is non-canonically isomorphic to $G_{\mathcal{M}_T}(\mathcal{O})$ *i.e.* is the motivic Galois group ([40], [36]) of the scheme $S_4 = \mathrm{Spec}(\mathcal{O})$ of 4-cyclotomic integers.

We show that there is a universal singular frame in which, when using dimensional regularization and the minimal subtraction scheme all divergences disappear. When computed as iterated integrals, the coefficients of the universal singular frame are certain rational numbers which are the same as the "mysterious" coefficients in the local index formula [25] of noncommutative geometry of section 3.4. This suggests deeper relations between the use of the renormalization group in the case of higher multiplicities in the dimension spectrum in the proof of the local index formula [25] and the theory of anomalies in chiral quantum field theories.

This work realizes the hope formulated in [23] of relating concretely the renormalization group to a Galois group and confirms a suggestion made by Cartier in [3], that in the Connes–Kreimer theory of perturbative renormalization one should find

[23]where $\mathbb{G}_m$ stands here for the multiplicative group $\mathbb{C}^*$.

a hidden "cosmic Galois group" closely related in structure to the Grothendieck–Teichmüller group.

These facts altogether indicate that the divergences of Quantum Field Theory, far from just being an unwanted nuisance, are a clear sign of the presence of totally unexpected symmetries of geometric origin. This shows, in particular, that one should try to understand how the universal singular frame "renormalizes" the geometry of space-time using the Dim-Reg minimal subtraction scheme and the universal counterterms. The next step is to combine this understanding with the above discussion of the standard model and apply it to the spectral action considered as a very specific quantum field theory dictated by experimental evidence.

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